Lecture 12. Electromotive force
Kirchhoff's rules

Last time:

* Electric current $I = \frac{dQ}{dt}$ in terms of current density $I = \int_S \underline{J} \cdot d\underline{a}$

* Electric current density

$$\underline{J} = q n \underline{v} = \rho \underline{v}$$

where n is the # charges / unit volume
$\underline{v}$ is the vel of charges
ρ is the resistivity

* Continuity equation

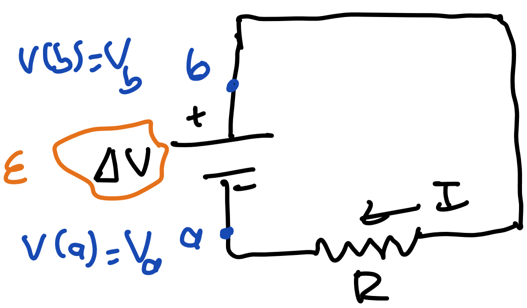
$$\nabla \cdot \underline{J} + \frac{\partial \rho}{\partial t} = 0$$ conservation of charge

* Ohm's law

$\underline{J} = \sigma \underline{E}$ microscopic where $\sigma = \frac{nq^2 \tau}{m}$ conductivity

$V = RI$ macroscopic where $R = \frac{L}{\sigma A}$ resistance

Today: EMF. Electromotive force (not a force)



Consider a circuit is made of a battery & a resistor w/ resistance R.

Let the potential difference be:

$$\Delta V = V_b - V_a > 0$$

If a charge Δq is moved from a to b, it's potential is increased

$$\Delta U = \Delta q \Delta V$$

As it moves through the resistor, it will dissipate energy, so upon returning to a, the potential energy of Δq is unchanged

The rate of energy loss through the resistor:

$$P = \frac{\Delta U}{\Delta t} = \left(\frac{\Delta q}{\Delta t}\right)\Delta V = I \Delta V$$ power supplied by

Δt $\quad I \quad (\Delta t)$ the voltage source

$\Downarrow$

battery

We need to supply energy to maintain a current. This source of energy is referred to as the EMF.

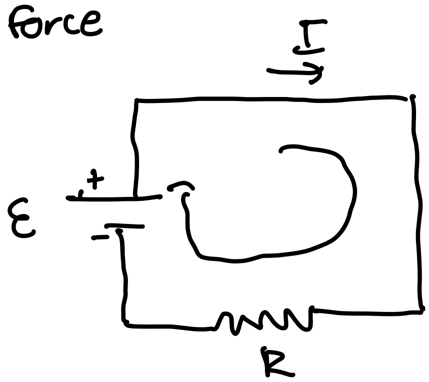
$$\boxed{\varepsilon = \frac{dW}{dq}}$$

work done to move a unit charge in the direction of higher potential

$[\varepsilon] = \text{Volt}$

Thoughts on EMF:

* We can think of it as a "charge pump" that moves charge from regions of lower potential to higher one.
* Batteries, solar cells, thermocouples are examples of sources of emf.
* Not a force



* Assuming none of the components have an internal resistance $\Delta V = \varepsilon$ emf
* When crossing from the negative to the positive terminal the potential increases by ε
* When crossing the resistor, the potential decreases by an amount IR

What happens to the potential as we go through the loop?

Upon completing a loop, the net change in potential difference is zero.

$\varepsilon - IR = 0$ Kirchoff's second rule

Around closed loops the sum of the emf + potential drops = 0

$W = -q\oint \vec{E}\cdot d\vec{s} = 0$ this is due to the conservative nature of the electrostatic field.

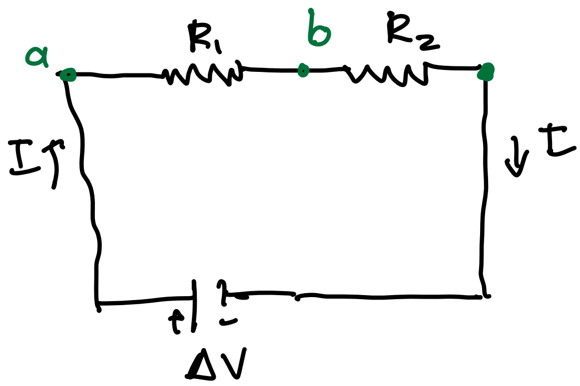
Important conventions regarding emf:

* $\overset{+}{\underset{-}{=}}$ this symbol indicates emf. Longer line is ⊕ terminal, shorter line is ⊖ terminal
* In circuits current flows from ⊕ to ⊖

Solving circuits.

Resistors in series and parallel

Solving circuits $\longrightarrow$ Determine the currents and voltages through all the elements



Two resistors R_1 and R_2 connected in series through a voltage source ΔV

The current traveling through both resistors is the same

The voltage drop from point a to point c is the sum of the voltage drops across individual resistors

$$\Delta V = IR_1 + IR_2 = I(R_1 + R_2) \quad *$$

The two resistors can be replaced by one eq. resistance R_{eq} :

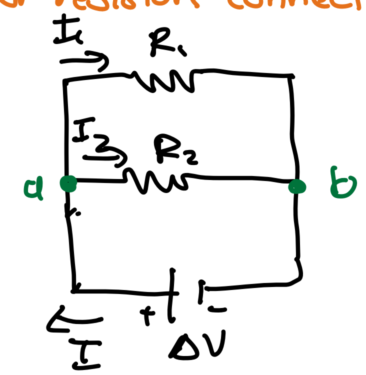
$$\Delta V = I R_{eq} \quad ** \quad \Rightarrow \boxed{R_{eq} = R_1 + R_2}$$

We can extend this argument to N resistors:

$$\boxed{R_{eq} = R_1 + R_2 + \dots = \sum_{i=1}^{N} R_i}$$

For resistors in series

For resistors connected in parallel:



* We have 2 resistors R_1 and R_2, connected in parallel across a voltage source ΔV

* The current I that passes through the voltage source is split into I_1 and I_2 that pass through R_1 and R_2

The potential across each resistor is the same :

$$\Delta V = \Delta V_1 = \Delta V_2$$

For each resistor we can write Ohm's law:

$$\Delta V = R_1 I_1 \qquad \Delta V = R_2 I_2$$

by current conservation

$$I = I_1 + I_2 = \frac{\Delta V}{R_1} + \frac{\Delta V}{R_2} = \Delta V \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

R_1 R_2

For 2 resistors in parallel we can replace R_1 and R_2 by the equivalent resistance:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

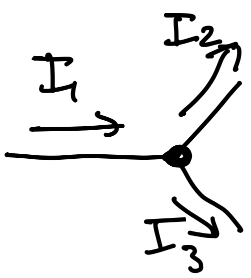
We can extend this for N resistors

$$\boxed{\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \dots = \sum_{i=1}^{N} \frac{1}{R_i}}$$

For a parallel connection

Kirchhoff's circuit rules

① Rule #1: Junction rule



$$\boxed{\sum I_{in} = \sum I_{out}}$$

For this example

$$I_1 = I_2 + I_3$$

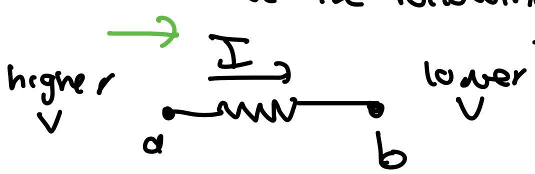
At any point where we have a junction the sum of the currents into the node must equal the sum of the currents out of the node

② Rule #2. Loop Rule

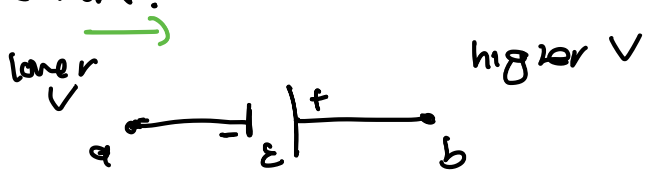
The sum of the voltage drops ΔV across circuit elements that form a closed circuit is zero

$$\boxed{\sum_{\substack{\text{closed} \\ \text{loop}}} \Delta V = 0}$$

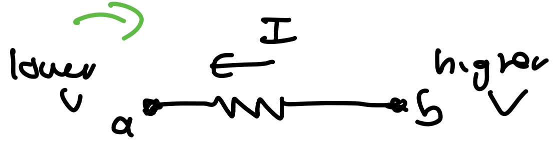
Using these 2 rules + Ohm's law we can solve any circuit. We can use the following conventions:



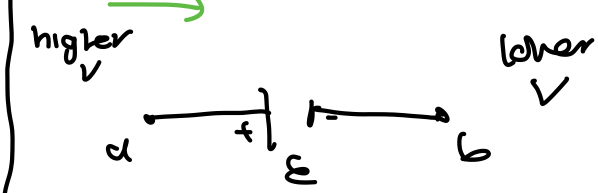
$$\Delta V = V_b - V_a = -IR$$



$$\Delta V = V_b - V_a = \mathcal{E}$$



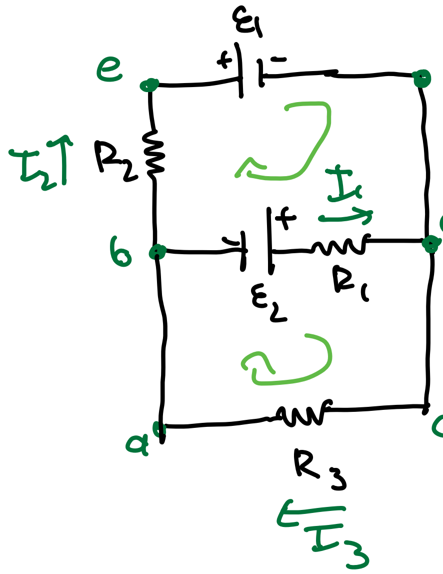
$$\Delta V = V_b - V_a = IR$$



$$\Delta V = V_b - V_a = -\mathcal{E}$$

→ indicates the travel direction

Example: A multiloop circuit



We have two sources of emf: $\mathcal{E}_1$ and $\mathcal{E}_2$

We know all the resistences: R_1, R_2, R_3

What is the current through each resistor?

We have 3 unknowns I_1, I_2, I_3 → we need 3 equations

First we apply Rule #1 to junction b:

For node b:

$$\underbrace{I_1 + I_2}_{\text{leaving}} = \underbrace{I_3}_{\text{incoming}}$$

To obtain 2 more equations we use the loop role: The net potential difference across a closed loop is zero.

Going across befc in a clockwise direction, the voltages are:

$$-I_2R_2 - \mathcal{E}_1 + I_1R_1 - \mathcal{E}_2 = 0$$

Traversing loop abcd clockwise yields:

$$\mathcal{E}_2 - I_1R_1 - I_3R_3 = 0$$

So the equations we need to solve are:

$$I_3 = I_1 + I_2$$

$$-I_2R_2 - \mathcal{E}_1 + I_1R_1 - \mathcal{E}_2 = 0$$

$$\mathcal{E}_2 - I_1R_1 - I_3R_3 = 0$$

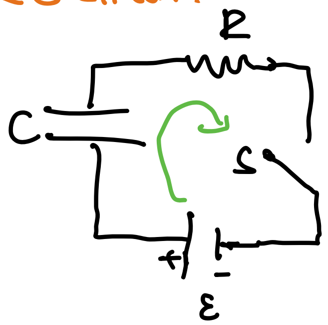
after doing algebra we get:

$$I_1 = \frac{\mathcal{E}_1R_3 + \mathcal{E}_2R_3 + \mathcal{E}_2R_2}{R_1R_2 + R_1R_3 + R_2R_3}, \quad I_2 = -\frac{\mathcal{E}_1R_1 + \mathcal{E}_1R_3 + \mathcal{E}_2R_3}{R_1R_2 + R_1R_3 + R_2R_3}$$

the direction of I_2 is opposite to what we assumed initially

$$I_3 = \frac{\varepsilon_2 R_2 - \varepsilon_1 R_1}{R_1R_2 + R_1R_3 + R_2R_3}$$

RC circuit



S ≡ switch
C ≡ capacitor
R ≡ resistor
ε ≡ emf source

Charging a capacitor

At $t < 0$ switch is open, there is no voltage across the capacitor

At $t = 0$ we close the switch & current begins to flow

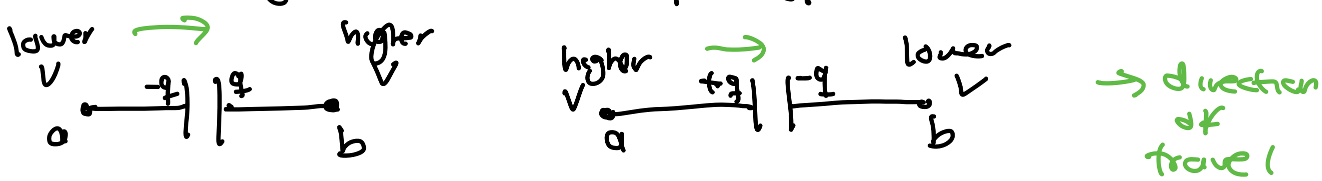
$$I_0 = \frac{\varepsilon}{R}$$

At this instant the potential difference from the battery terminals is the same as across the resistor.
→ this will initiate charging of the capacitor

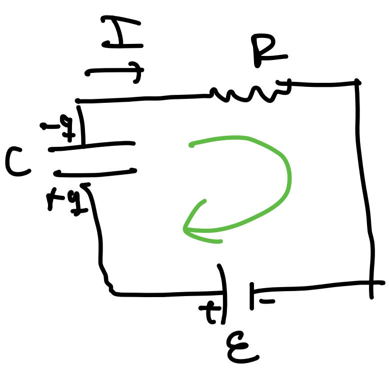
As the capacitor charges, Voltage across it increases in time

$$V_c(t) = \frac{q(t)}{C} \qquad \left(\text{Remember } C = \frac{Q}{V}\right)$$

How much charge is stored in the plates?



$$\Delta V = V_b - V_a = \frac{q}{C} \qquad \Delta V = V_b - V_a = -\frac{q}{C}$$



Apply loop rule in a clockwise direction:

$$\varepsilon + \underbrace{V_c(t)}_{\text{Voltage across capacitor}} - \underbrace{I(t)R}_{\text{Voltage across resistor}} = 0 \quad (*)$$

$$I = \frac{dq}{dt}$$

$$\Rightarrow \varepsilon - \frac{q}{C} - \frac{dq}{dt}R = 0 \quad (**)$$

Over time capacitor will charge until it reaches a maximum value Q. At this point $I \to 0$ From (*)

$I(t)R = \varepsilon - V_c(t)$ once V_c reaches ε → current is zero.

Eq (**) gives us a relation between the rate of change of charge and charge in capacitor:

$$\frac{dq}{dt} = \frac{1}{R}\left(\varepsilon - \frac{q}{c}\right)$$

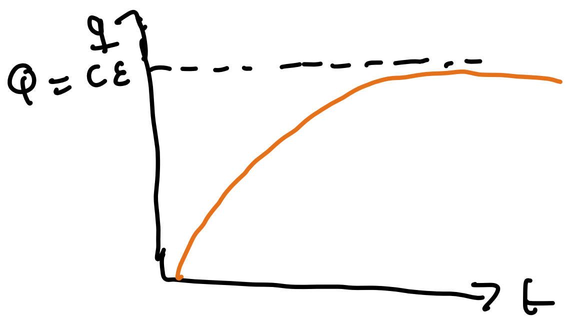
$$\Rightarrow \frac{dq}{q - C\varepsilon} = -\frac{1}{RC}\,dt$$ integrating both sides:

$$\int_0^q \frac{dq}{q - C\varepsilon} = -\frac{1}{RC}\int_0^t dt$$

$$\Rightarrow \ln\left(\frac{q - C\varepsilon}{-C\varepsilon}\right) = -\frac{t}{RC}$$

$$\Rightarrow \boxed{q(t) = C\varepsilon\left(1 - e^{-t/RC}\right) = Q\left(1 - e^{-t/RC}\right)}$$

where Q is the max charge stored in plates



$\tau = RC$ time constant
measure of the rise of the exponential

Once we know the charge on the capacitor we can look at how voltage changes across the capacitor:

$$V_c(t) = \frac{q(t)}{C} = \varepsilon\left(1 - e^{-t/RC}\right)$$

Now the current that flows through the circuit can be obtained from our expression for $q(t)$: $I = \frac{dq}{dt}$

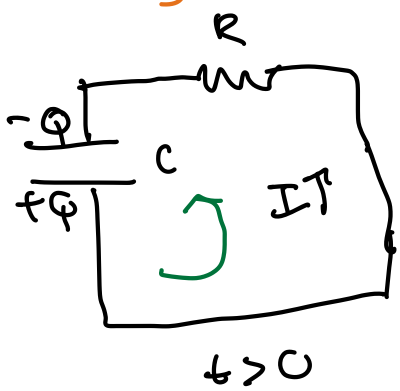
$$I = \frac{dq}{dt} = \frac{C\varepsilon}{RC}\left(e^{-t/RC}\right) = \frac{\varepsilon}{R}\,e^{-t/RC} = I_0 e^{-t/RC} = I_0 e^{-t/\tau}$$

I_0 initial current flow in the circuit (at $t=0$)

The current as a function of time:

current decreases exponentially with a time constant $\tau = RC$

Discharging a capacitor



- The capacitor was already charged to a value Q
- The switch at $t<0$ is open

What happens when we close the switch?

Applying the loop rule traversing the circuit counterclockwise we get:

$$\frac{q}{C} - IR = 0$$, we know $I = \frac{dq}{dt}$ so this eq can be rewritten:

$$\frac{q}{C} + R\frac{dq}{dt} = 0$$ We can solve this by integrating:

$$\frac{dq}{q} = -\frac{1}{RC}dt \Rightarrow \int_Q^q \frac{dq}{q} = -\frac{1}{RC}\int_0^t dt$$

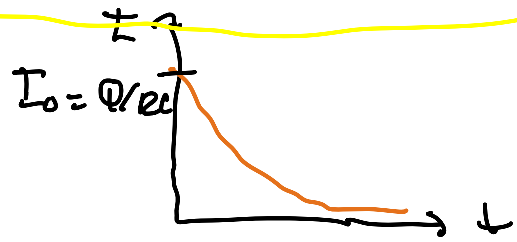
$$\Rightarrow \ln\left(\frac{q}{Q}\right) = -\frac{t}{RC} \Rightarrow \boxed{q(t) = Q e^{-t/RC}}$$

We can determine the voltage drop across the capacitor:

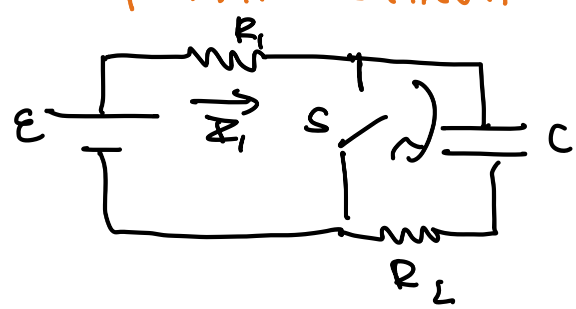
$$V_C(t) = \frac{q(t)}{C} = \left(\frac{Q}{C}\right)e^{-t/RC}$$

From the expression for $q(t)$ we can determine the current:

$$\boxed{I = -\frac{dq}{dt} = \left(\frac{Q}{RC}\right) e^{-t/RC}}$$



Example: An RC circuit



a) What is the time constant of the circuit before the switch is closed?
b) What is the time constant after the switch is closed?
c) Find the current through the switch after its closed?

a) Before the switch is closed, the 2 resistors R_1 and R_2 are in series with the capacitor. By definition the time constant is:

$$\tau = R_{eq}\, C = (R_1 + R_2)\, C$$

the charge stored in the capacitor depends on this time constant as:

$$q(t) = C\varepsilon\left(1 - e^{-t/\tau}\right)$$

b) After the switch is close, the closed loop on the right becomes a decaying RC circuit, with time constant $\tau^* = R_2 C$

charge will begin to decay according:

$$q^*(t) = C\varepsilon\, e^{-t/\tau^*} \leftarrow$$

c) The current passing through the switch is from 2 source's:

Steady current I_1 and a decaying current I_2

$$I_1 = \frac{\varepsilon}{R_1}$$

$$I_2 = \frac{dq^*}{dt} = -\left(\frac{C\varepsilon}{\tau^*}\right)e^{-t/\tau^*} = -\left(\frac{\varepsilon}{R_2}\right)e^{-t/R_2C}$$

flows in the opposite direction of charging.

The total current will be:

$$\boxed{I(t) = I_1 + I_2(t) = \frac{\varepsilon}{R_1} + \left(\frac{\varepsilon}{R_2}\right) e^{-t/R_2C}}$$